Aboutness, Quantification, and λ-conversion

1 Introduction

There's a sense in which universal claims are about the entities they quantify over (call this A). For example, 'every proposition is true' is about all propositions, and 'everyone will eventually die' is about, well, everyone. On the other hand, existential claims don't seem to follow this rule. That is, they're not always about everything they quantify over; they're only about the things that are true of (call this E): thus 'someone called me today' is clearly not about everyone—it's about the one(s) who called me today; nor is 'some sets have two members' about all sets —it's only about sets that have two elements. Add to these a further plausible, and in fact widely accepted assumption about the interaction of aboutness and negation: if ϕ is about a, then so is $\neg\phi$ (call this N).

These principles altogether imply that existential statements don't express propositions that are identical to the dual propositions expressed by universal statements and negation. That is, for example, the proposition that something is F isn't identical to the proposition that not everything is a non-F—at best they are materially equivalent. The argument is straightforward: the latter proposition, due to N and A is about everything, whereas the former, due to E, isn't. So they don't share the same properties, hence cannot be the same.

In this paper, I will propose a rigorous treatment of the informal argument above, in the rich language of higher-order logic. I will, in particular, argue against (i) outright identifying existential claims with their dual universal claims:

e.g., $\exists x^t \phi = \neg\forall x^t \neg\phi$, and (ii) defining higher-order existential quantifier in terms λ-abstraction, universal quantification and negation (i.e., $\exists^t := \lambda X^{\langle t \rangle}.\neg\forall x^t \neg X(x)$, where $\exists^t$ is construed as higher-order existential quantifier) in the context of higher-order logic equipped with the principle λ-conversion, also known as β.

2 Interdefinability of the Logical Vocabulary

Identifying dual statements is a move that logicians often take, in general, about dualities in logic. Examples of this sort are $\phi \wedge \psi = \neg(\neg\phi \vee \neg\psi)$, $\phi \rightarrow \psi = \neg\phi \vee \psi$, and $\exists x\phi := \neg\forall x\neg\phi$, where, say, the operators disjunction and negation, as well as the universal quantification are taken as primitives, and the rest are defined in terms of them. Call this *reductionism.* For the logician, the immediate advantage reductionism is its convenience: for example, one no more has to worry about the tedious task of checking a desired feature of formulas for every connective, which is what they'd have to do if all connectives and operators were introduced primitively instead of interdefinably. Instead, they can just check if the desired property holds for formulas made of the smaller selection of connectives and/or operators, and generalize the results to the interdefined ones. In short: for most formal purposes not only there's no drawback in identifying dual statements, but it makes things much easier.

When it comes to metaphysics, however, reductionism can have significant impacts on one's outlook. For instance, according to some natural views about propositional granularity, two equivalent propositions may well be distinct, so it's no longer obvious that, say, $\neg(\phi \rightarrow \neg\psi)$ and $\neg(\neg\phi \vee \neg\psi)$ express the same propositions, assuming the connectives used are the actual well-known Boolean ones. So it's no longer obvious which one of these, if any, should be construed as $\phi \wedge \psi$. To decide one over another on an arbitary basis would be to prejudge matters of granularity. Moreover, such identifications can lead to inconsistencies in certain granular theories, such as the theory of immediate ground (Wilhelm 2020); we'll get back to this point in §5.

The most neutral option is, then, to take every connective and operator as a primitive, and propose their desired features in terms of the axioms of a proof theory that introduces their behavior. Call this position *radical primitivism.* But radical primitivism, even though metaphysically promissing, makes life difficult for practical purposes: e.g., every proof that calls for a simple induction over the structure of terms would have to go through each and every connective and operator and check if the desired feature holds.

A third, somewhat smarter option is to take some connectives and operators as primitives, and interdefine a set of other connectives and operators in terms of them—as if they were the other actual logical vocabulary—but not necessarily consider the new vocabulary as the actual, missing logical ones. Call this position *modest primitivism.* For example, suppose we take our primitive connectives and quantifiers to be disjunction ($\vee$), negation ($\neg$) and universal quantifier ($\forall x\phi$), and define some other connectives and quantfiers in terms of these, in the usual way, but stay neutral regaring whether they actually are the other, know logical operators. For example, let $\phi \wedge' \psi := \neg\phi \vee \neg\psi$, and call this sunjunction. The neutralist stays neutral about whether sunjunction is the same as conjunction. They have the same truth conditions, and for most purposes that's all we want from either of them, they well may be different things. In higher-order logic, neutralism can also be implemented with the help of λ-abstraction. For example, suppose we take our primitive logical constants to be disjunction ($\vee$), negation ($\neg$) and higher-order universal quantifier ($\forall^t$, for each type t). Now, we can again define sunjunction, but now with the help of λ-abstraction: $\wedge' := \lambda x \lambda y.\neg(\neg\phi\vee\neg\psi)$. Similarly, one could define $E^t := \lambda X^{\langle t\rangle}.\neg\forall x^t \neg X(x)$, and remain neutral about whether or not this represents existential quantification.

3 Aboutness in Higher-Order Logic

Introduction to higher-order logic

To frame the problem of structured propositions with sufficient rigor, allow me to introduce a simple type theory (STT).

Types provide a way to track the grammatical categories of expressions.

Definition 1 (Types)**.** The set $\mathcal{T}$ of *types* is recursively defined as follows: $e \in \mathcal{T}$, $\langle\rangle \in \mathcal{T}$, and for any types $t_1, ..., t_n$, $\langle t_1, ..., t_n\rangle \in \mathcal{T}$.

Before defining terms of the system, we assume that for any $t \in \mathcal{T}$ there's a denumerably infinite set of *variables* Var^t of type t and a (possibly empty) set of typed non-logical *constants* CST^t. For certain types there are also logical constants to be introduced below. (We will reserve CST^t for the set of all constants (logical or non-logical) of type t.) We define the sets of all variables and constants respectively as $\mathrm{Var} := \bigcup_{t\in\mathcal{T}^r} \mathrm{Var}^t$ and $\mathrm{CST} := \bigcup_{t\in\mathcal{T}^r} \mathrm{CST}^t$.

Treating the logical vocabulary as constants is the prevalent approach in higher-order logic.[1] Here's the list of our primitive, typed logical constants: implication, $\rightarrow$, of type $\langle\langle\rangle,\langle\rangle\rangle$, and for any type t, there is a constant for a (higher-order) universal quantifier $\forall^t$, of type $\langle\langle t\rangle\rangle$, existential quantifier $\exists^t$, of type $\langle\langle t\rangle\rangle$ and identity $=^t$ of type $\langle t, t\rangle$. After we introduce the set of terms of STT, we will see how the quantifiers function with the given type, and how other connectives will be defined in terms of the primitive constants above. Call the language with the logical constants above $\mathcal{L}$.

Definition 2 (Terms of $\mathcal{L}$)**.** The *terms* in $\mathcal{L}$ are recursively defined as follows: **(i)** if x is a variable of type t, then x is a term of type t**;** **(ii)** if c is a constant of type t, then c is a term of type t**;** **(iii)** if ϕ is a terms of type $\langle\rangle$ and for $n \geq 1$, the variables $x_1, ..., x_n$ are pairwise distinct, and respectively of types $t_1, \ldots, t_n$, then

[1] For example, see (Bacon 2018; Bacon 2019; Church 1940; Dorr 2016; Dorr et al. MS; Henkin 1950; Mitchell 1996).

$\lambda x_1^{t_1},...,x_n^{t_n}.\phi$ is a term of type $\langle t_1,...,t_n \rangle$ **; (iv)** if τ is a term of type $\langle t_1,...,t_n \rangle$, where $n \geq 1$, and for each $i = 1,...,n$, σ_i is a term of type t_i, then $\tau(\sigma_1,...,\sigma_n)$ is a term of type $\langle\rangle$.

We also have the following set of interdefined connectives and operators:

$\bot := (\forall^{\langle\rangle})(\lambda p^{\langle\rangle}.p)$	$\neg := \lambda p^{\langle\rangle}.(p \to \bot)$	$\vee := \lambda p^{\langle\rangle} q^{\langle\rangle}.(\neg p \to q)$
$\wedge := \lambda p^{\langle\rangle} q^{\langle\rangle}.\neg(p \to \neg q)$		$=^t := \lambda x^t y^t.\forall^{\langle t \rangle} X(X(x) \to X(y))$

For our purposes, it doesn't matter if we are reductionist or modest primitivist about these connectives, although in the end we will get back to this matter. We call a term of type $\langle\rangle$ a *formula*, and when it contains no free variables, a *sentence*. We use the letter t with or without subscripts as metavariables for types, lower-case Greek letters $\tau, \sigma, \phi, \psi, ...$ with or without subscripts as metavariables for general terms, and lower-case or capital English letters $x, y, z, p, q, X, Y, Z, P, Q$, with or without subscripts, as metavariables for variables. Also, from now on, by convention, we write things like $\phi \vee \psi$ or $x = y$ to indicate the application instances $\vee(\phi, \psi)$ or $=(x, y)$, and so on. The notions of *free* and *bound* variables of terms, substitutions of terms for variables, and *being free for a variable*, are defined as usual. We show the set of free variables in a term σ by $FV(\sigma)$. Also the set of all terms of $\mathcal{L}$ is denoted by TERM.

Now, we will propose a proof theory for our language $\mathcal{L}$.

System $\mathcal{PH}$

Axioms:

1. All theorems of propositional logic. PL

2. $\vdash (\lambda x_1^{t_1},...,x_n^{t_n}.\phi)(\sigma_1,...,\sigma_n) = [\sigma_1/x_1,...,\sigma_n/x_n]\phi$, where the type of σ_i is t_i, for each $i = 1,...,n$. β[2]

[2]Although this won't impact our arguments going forward, it's worth pointing out that some people might want to restrict λ-abstraction to non-vacuous cases only, because otherwise some non-trivial consequences may follow regarding aboutness: for example, by β we have:

3. $\vdash \forall^t F \to F(\sigma)$, where F is of type $\langle t \rangle$, and the type of σ is t. UI

4. $\vdash F(\sigma) \to \exists^t F$, where F is of type $\langle t \rangle$, and the type of σ is t . EG

5. $\vdash \forall^t(\lambda x^t.\ \phi \to F(x)) \to (\phi \to \forall^t F)$, where $x \notin FV(\phi)$ UD

6. $\vdash \forall^t(\lambda x^t.\ F(x) \to \psi) \to (\exists^t F \to \psi)$, where $x \notin FV(\psi)$ ED

Rules of Inference:

7. If $\vdash \phi$ and $\vdash \phi \to \psi$, then $\vdash \psi$. MP

8. If $\vdash F(x)$, then $\vdash \forall^t(F)$, where x is of type t, and F is a variable of type $\langle t \rangle$. GEN

Now, consider the language which has all the variables and non-logical constants that $\mathcal{L}$ has, except that it doesn't have the existential quantifier $\exists^t$ as a primitive constant, but as the combinator $\exists^t := \lambda X^{\langle t \rangle}.\neg\forall x^t \neg X(x)$. Call this language $\mathcal{L}^-$. Also, leave the term-formation rules as they are. Then the axioms EG and ED can be derived as the theorems of the rest of the axioms plus the derivation rules of $\mathcal{PH}$. Call the shrunk proof system $\mathcal{PH}^-$.

Theorem 1. *$\vdash^{\mathcal{L}}_{\mathcal{PH}} \exists^t(F) \leftrightarrow \neg\forall^t(\lambda x^t.\neg F(x))$, where F is type $\langle t \rangle$.*

Proof.

(λx^e.The Sun is shining)(Amir) = The Sun is shining; but the left-side of the identity is presumably a aproposition about Amir, whearas the the right side clearly isn't (Dorr 2016). This problem, however, doesn't seem to arise if we replace β with β_E, and leave vacuous cases of abstraction intact, because materially equivalent propositions aren't necessarly about everything: for example *2+2=5* and *The Earth is flat* are equivalent, but they're not about the same things.

(1) $\vdash \forall^t(\lambda x^t.\ F(x){\to}\bot) \to (\exists^t F \to \bot)$ ED

(2) $\vdash \forall^t(\lambda x^t.\ \neg F(x)) \to \neg\exists^t(F)$ β 1; $\neg := \lambda p^{\langle\rangle}.(p \to \bot)$

(3) $\vdash \exists^t(F) \to \neg\forall^t(\lambda x^t.\ \neg F(x))$ PL 3

(4) $\vdash F(x) \to \exists^t(F)$. EG

(5) $\vdash \neg\exists^t(F) \to \neg F(x)$ PL 4

(6) $\vdash \neg\exists^t(F) \to\ (\lambda x^t.\neg F(x))(x)$ β 5

(7) $\vdash (\lambda x^t.\neg F(x))(x) \to \forall^t(\lambda x^t.\neg F(x))$ GEN

(8) $\vdash \neg\exists^t(F) \to \forall^t(\lambda x^t.\neg F(x))$ PL 6, 7

(9) $\vdash \neg\forall^t(\lambda x^t.\neg F(x)) \to \exists^t F$ PL 8

(10) $\vdash \exists^t(F) \leftrightarrow \neg\forall^t(\lambda x^t.\neg F(x))$ PL 3, 9 □

Remark 1. Notice that the proof would still go through if we replace β with the weaker β_E, and add (2) as an axiom for a primitive $\neg$ of our system. In particular, in a system where all of the logical vocabulary are given as primitives with appropriate axiomatization, the result above still holds even with β_E in place of β.

Theorem 2. *$\vdash^{\mathcal{L}^-}_{\mathcal{PH}^-} \exists^t(F) \leftrightarrow \neg\forall^t(\lambda x^t.\neg F(x))$, where F is type $\langle t\rangle$.*

Proof. Straightforward. Notice that the proof would go through even if we replace β with β_E. □

Remark 2. Notice that, in either of the theorems above, proving identities of the two sides of $\leftrightarrow$ needs something like an axiom of extensionality for propositions—$\forall p^{\langle\rangle}\forall q^{\langle\rangle}(p{\leftrightarrow}q \to p{=}q)$—which we haven't adopted in our system.

Finally, the following (schematic) lemma will come in handy:

Lemma 1 (LBZ). *$\sigma =^t \tau \to \forall X^{\langle t\langle\rangle\rangle}(X(\sigma) \leftrightarrow X(\tau))$.*

3.1 Aboutness in Higher-order Logic

Now we formalize our principles of aboutness from Introduction. For any terms ϕ of type $\langle\rangle$ and σ of any type t, let $A(\phi, \sigma)$ be understood as 'ϕ is about σ'. So, for any type t, we can take A to be constant of type $\langle\langle\rangle, t\rangle$. Here's the list of the informal assumptions from Introduction, expressed in the language of higher-order logic with the following schemata:

$$\text{(A)} \qquad \forall y^t A(\forall x^t \phi, y).$$

$$\text{(E)} \qquad \forall y^t([y/x]\phi \leftrightarrow A(\exists x^t \phi, y)).$$

$$\text{(N)} \qquad \forall y^t(A(\phi, y) \rightarrow A(\neg\phi, y)).$$

Of course, one might be able to derive these from a more general account of (higher-order) aboutness, but here we are not concerned with any such account, and we just take at face value these as theorems of our minimal theory of aboutness, which, as argued briefly, seem to conform to our intuitions about the interaction of aboutness and quantification.

Notice that since the principles above are schematic in the types t and formulas ϕ, we can instantiate them with different types and formulas. In particular, let $t \equiv \langle\rangle$. Since provably some propositions are false (otherwise the systems $\mathcal{PH}$ and $\mathcal{PH}^-$ will both be inconsistent, which they aren't), the there proposition $\exists p^{\langle\rangle} p$ isn't about all propositions. As a result we have the following instances of the rules above:

$$\text{(A}'\text{)} \qquad \forall q^{\langle\rangle} A(\forall p^{\langle\rangle} \neg p, q).$$

$$\text{(E}'\text{)} \qquad \neg\forall q^{\langle\rangle} A(\exists p^{\langle\rangle} p, q).$$

$$\text{(N}'\text{)} \qquad \forall q^{\langle\rangle}(A(\forall p^{\langle\rangle} \neg p, q) \rightarrow A(\neg\forall p^{\langle\rangle} \neg p, q)).$$

3.2 The Aboutness Argument in Higher-Order Logic

It's an easy task to see that identifications of the form $\exists x^t \phi = \neg\forall x^t \neg\phi$, where $\exists x^t \phi$ is construed as existential quantification (which correspond to the first strategy glossed in Section 2) lead to inconsistency with $\{\mathrm{A}', \mathrm{E}', \mathrm{N}'\}$ (hence, with $\{\mathrm{A}, \mathrm{E}, \mathrm{N}\}$), in higher-order logic.[3] Suppose $\mathcal{L}^A$ is the enrichment of $\mathcal{L}$ with aboutness constants A of different types. (Similarly, $\mathcal{L}^{-A}$ is just $\mathcal{L}^-$ with those constants added.)

Theorem 3. $\{\mathrm{A}, \mathrm{E}, \mathrm{N}\} \nvdash^{\mathcal{L}^{\mathrm{A}}}_{\mathcal{PH}} \bot$.

Proof. The proof of this theorem needs a kind of model theory that's beyond the scope of the paper. □

Remark 3. The reason that equivalence statements like $\exists x^t \phi \leftrightarrow \neg\forall x^t \neg\phi$ (or alternatively, replacing β with β_E, in $\mathcal{PH}^-$) don't lead to inconsistency with our minimal principles of aboutness (Theorem 3) must have to do with this observation: in general from $A(\phi, a)$ and $\phi \to \psi$ we cannot conclude $A(\psi, a)$. For example, by A we have it that the proposition $\forall p^{\langle\rangle}(p = p)$ is about every proposition, and by UI we have $\forall p^{\langle\rangle}(p = p) \to \bot = \bot$, but clearly the latter isn't about every proposition. This can be introduced as a separate principle, or might follow from some other, more basic, principles.

Theorem 4. $\{\mathrm{A}', \mathrm{E}', \mathrm{N}'\} \vdash^{\mathcal{L}^{-\mathrm{A}}}_{\mathcal{PH}^-} \bot$.

Proof.

[3] This can straightforwardly also be proved for weaker logics such as first-order logic, as well. So in such logics, one would need to take existential statements as primitives, in order to avoid inconsistency.

(1) $\forall q^{\langle\rangle} A(\forall p^{\langle\rangle} \neg p, q)$ A′

(2) $A(\forall p^{\langle\rangle} \neg p, q)$ UI 1

(3) $A(\neg\forall p^{\langle\rangle} \neg p, q)$ N′ 2

(4) $\neg\forall p^{\langle\rangle} \neg p = (\lambda X^{\langle\langle\rangle\rangle}.\neg\forall p^{\langle\rangle} \neg X(p))(\lambda r^{\langle\rangle}.r)$ β

(5) $A\big((\lambda X^{\langle\langle\rangle\rangle}.\neg\forall p^{\langle\rangle} \neg X(p))(\lambda r^{\langle\rangle}.r), q\big)$ LBZ 3, 4

(6) $\forall q^{\langle\rangle} A\big((\lambda X^{\langle\langle\rangle\rangle}.\neg\forall p^{\langle\rangle} \neg X(p))(\lambda r^{\langle\rangle}.r), q\big)$ GEN 5

(7) $\forall q^{\langle\rangle} A\big(\exists^{\langle\rangle}(\lambda r^{\langle\rangle}.r), q\big)$ $\exists^t := \lambda X^{\langle t\rangle}.\neg\forall x^t \neg X(x)$

(8) $\bot$ E′ 7 $\square$

As a result, we also have: $\{\mathrm{A}, \mathrm{E}, \mathrm{N}\} \vdash^{\mathcal{L}^{-\mathrm{A}}}_{\mathcal{PH}^{-}} \bot$.

4 Conclusion

The main result of this paper, which is to reject reductionism, gainst further support from the literature on grounding. For example, Wilhelm (2020) has shown (although, for simpler languages than that of higher-order logic) identifications of the form $\phi \wedge \psi = \neg(\neg\phi \vee \neg\psi)$, where the connectives invloved are conjunction and disjunction, respectively, also lead to inconsistency with the standard principles of immediate ground. Of course, one way to interpret wilhelm's results is that the notion of immediate ground is unitelligible. But I would contend that a more wholesome interpretation is to reject reductionism, because (i) as was dicsussed earlier, this leads to prejudgment of matters of granularity, and (ii) considerations that underly identifications of the form $\phi \wedge \psi = \neg(\neg\phi \vee \neg\psi)$ are similar to the ones of the form the form $\exists x^t \phi = \neg\forall x^t \neg\phi$; so if we go against the latter, it would sound arbitrary not to do to the same for the former. That is, either of radical or modest primitivism seem more metaphysically promissing than reductionism, although the modest primitivist, much like the reductionist, has the disadvantage of having to awkwardly choose the 'elite' collection of primitive operators, as it's not clear on what basis a select group of operators be chosen over another.

In higher-order languages where λ-abstraction is available the matter becomes more delicate, as one still could be reductionist to some extent, and there can be some advantages in that. In fact, one can pinpoint a specific principle in $\mathcal{PH}$ or $\mathcal{PH}^-$ that contributed to the inconsistency (Theorem 4), namely β: reductionism will be compatible with our minimal principles of aboutness if β is dropped. Notice also that, in such languages Wilhelm (2020)'s inconsistency result can be understaood as, for instance, defining conjunction as $\wedge := \lambda p^{\langle\rangle} q^{\langle\rangle}.\neg(\neg p \vee \neg q)$, where $\vee$ is disjunction; such identifications are inconsistent with the logic of immediate ground only if β holds.

Furthermore, recently Fritz (2020) has suggested to resolve a famous puzzle of mediate ground due to Fine (2010) and Krämer (2013) by replacing the first-order (or second-order) existential quantifiers with the *higher-order* existential quantification $\exists^t$. Fritz, in particular, shows that the puzzle is misguided if one rejects β (as well as another principle about the interaction of ground and λ-abstraction), and in the rest of his paper attempts to motivate such a move. My argument provides an independent reason for doing so: if we take $\exists^t$ to just be $\lambda X^{\langle t\rangle}.\neg\forall x^t \neg X(x)$, then $\exists^t$ *cannot* be the existential quantifier if β holds, so construing existence in the original puzzle as the higher-order operator $\exists^t$ won't be appropriate, under the rein of β, to begin with. Yet another ground-frinedly support for dropping β comes from a puzzle about the grounds of λ-abstractions, due to Fine (2012). As Dorr (2016) mentions, the puzzle won't go through if β is dropped.

Therefore, as long as one theorizes about metaphysical matters in the expressive language of higher-order logic, a rival to to radical primitivism and neutralism comes to the surface: reductionism without β, or *β-free reductionism.* The choice between these alternatives isn't obvious anymore: the latter is in particular more friendly to the notion of ground, contributing to solving two of its puzzles and securing it coherence.

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