CANTOR, GOD, AND INCONSISTENT MULTIPLICITIES

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The Theological acceptance of his set theory was very important to Cantor. Despite this, the influence of theology on his conception of absolutely infinite collections, or inconsistent multiplicities, is largely ignored in the contemporary literature. I will be arguing that due in part to his religious convictions, and despite an apparent tension between his earlier and later writings, Cantor would never have considered inconsistent multiplicities (similar to what we now call proper classes) as completed in a *mathematical* sense, though they are completed in *Intellectus Divino*.

Before delving into the issue of the actuality or otherwise of certain collections, it will be informative to give an explanation of Cantor's terminology, as well a sketch of Cantor's relationship with religion and religious figures. Such will comprise the first part of this paper, after which I will argue that although there is tension between how Cantor discusses the absolute infinite before roughly 1896, and inconsistent multiplicities after, due to his continuing and even strengthening religious convictions, Cantor would have maintained his earlier position that inconsistent multiplicities are not *mathematically* extant, and also not merely potential. I achieve this aim by first pointing out that the evidence taken by Jané (1995) to show that Cantor changed his view on absolute infinities is in fact consistent with my opposing thesis, and then pointing to additional evidence from Cantor's later writings that supports my view.

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1. Cantor and Religion

Interesting, and also important to Cantor's way of thinking is the fact that, despite his copious correspondence with Catholic theologians (cf. Tapp, 2005), he was in fact Protestant. His familiarity with Catholicism was due in large part to his reading of Aquinas, as well as his mother's Catholicism. Perhaps because of the tension between Catholicism and Protestantism in Cantor's life, he was not adherent to any of the organised churches, and thus was comfortable questioning Catholic dogma. Further evidence of Cantor's willingness to question Catholic teaching is his publication, in 1905, of a pamphlet entitled *Ex oriente lux*, in which he argues that Joseph of Arimathea was the biological father of Jesus, thereby denying the virgin birth (Tapp, 2005, §6.9). All of this is not to say that Cantor wasn't religious, but more on this in §6.

Despite his willingness to question the Church, it was very important to Cantor to show that his theory of actually infinite sets could be rectified with Catholic teaching which traditionally held that the only completed infinite was the infinite of God. This may have been partly a result of Cantor's apparent belief that set theory was given to him directly by God. This belief is evidenced by letters to Gösta Mittag-Leffler from the winter of 1883–4 in which Cantor claimed explicitly to have been given the content of his articles by God, having only provided the organisation and style himself (see Dauben, 1990, p. 146). Further evidence of Cantor's perceived connection to God comes from a letter to his father early in his mathematical career where he speaks of an "unknown, secret voice [compelling] him to study mathematics" (Dauben, 1990, p. 288); and a letter to Hermite in 1894 in which he thanks God for constraining him to Halle (by denying him a position in Göttingen or Berlin) so that he could better serve Him and the Catholic Church (Dauben, 1990; Meschkowski, 1967).

Combined with the publication of *Ex oriente Lux*, the letter to Hermite provides good evidence outside of Cantor's mathematical correspondence with theologians, of his continued, or even deepening connection to God and the Catholic church later in his career. Additionally, it is worth noting that the vast majority of Cantor's theological correspondence was written after his first psychological breakdown of 1884. In fact Dauben (1990) and Meschkowski (1967) hint a an explicit connection between Cantor's declining mental health and his increased interest in religion. The correlation may also be explained in part by the lack of non-religious reading material available to him during his incapacitations (Meschkowski, 1967)

2. Metaphysics of Sets

When thinking about Cantor's philosophy and theology of mathematics it is important to remember that Cantor was a platonist about mathematics, which is to say a realist about independently existent mathematical objects. This is not the platonism of Frege of Gödel however, as Cantor's plantonism is based on a kind of theological psychologism. He believed that any coherent mathematical object (mathematical objects quantified over in a coherent mathematical system) must exist necessarily due to God's omniscience, omnipotence and magnificence. The reasoning is that any coherent mathematical object possibly exists, and every possibly existent object *already* exists in the mind of God, thus, due to the above mentioned qualities of God, any coherent mathematical object exists.

Note here that Cantor is not precise about what he means by coherence, though intra-theoretic consistency seems to be a necessary, but likely not sufficient, condition.

What this means for us is that for Cantor, all of the cardinals and ordinals exist independently of mortal minds, and therefore there is a meaningful way in which the $\aleph$–sequence (say) is completed. I will argue below however, that the completion of such sequences, alternatively, the existence of what are now called proper classes, was to not to be considered a *mathematical* completion, but rather metaphysical and/or theological.

3. Actual, Potential, and Absolute Infinities

In his *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* (Foundations of a General Theory of Manifolds, hereafter *Grundlagen*) (Cantor, 1966, Nr. 5, originally published in 1883), Cantor makes the distinction between *Eigentlich-* and *Uneigentlichunendlichen*, usually translated as "proper" and "improper infinities". Later, in *Über die verschiedenen Standpunkte in bezug auf das aktuelle Unendliche* (published in 1886), Cantor identifies these with actual and potential infinities respectively (Tapp, 2005, §3). This essentially boils down to the difference between increasable and un-increasable infinities. It turns out however, that due to an idea of Cantor's that Hallett (1984) calls the "domain principle," which says that any potential or increasable series must have a domain into which it increases, the only actual infinite is the completed infinite. In other words, potential infinities may be heuristically useful, but are not actual.

A second distinction made by Cantor is between two kinds of actual infinities: the transfinite and the absolute. It is this distinction that will be central to the rest of the paper, as it is, in a sense at least, the distinction between the mathematical and the divine. The transfinite can be likened to domains of discourse —what potential infinities lead to, or increase into, while the absolute is the domain of God, embracing both the finite and the transfinite, and hence is unknowable. Cantor puts it like this: "the absolute can be acknowledged, but never known, nor approximately known" (Tapp, 2012, pp. 10–11).

It was the explication of this distinction that convinced many Catholic theologians, in particular Cardinal Franzelin, a papal theologian to the Vatican Council who was initially worried about the threat of pantheism often raised by theologians with respect to the actual infinite, that transfinite set theory was not a threat to Catholic doctrine (Dauben, 1990, p. 145).

4. (In)Consistent Multiplicities

4.1. **Epistolary Evidence.** Beginning around 1896, Cantor makes a terminological shift from writing about the absolute as opposed to the transfinite, and begins instead writing of consistent and inconsistent multiplicities. These were to be identified with sets and the absolute, respectively. Cantor introduces inconsistent multiplicities in his July 1899 letter to Dedekind thus:

> A multiplicity can be such that the assumption of the "togetherness" of *all* of its elements leads to a contradiction, so that it is impossible to regard the multiplicity as a unity, "a completed thing". Such collections I call *absolute infinities* or *inconsistent multiplicities*[all emphasis original](Meschkowski & Nilson, 1991, p. 407).

A paragraph later he defines consistent multiplicities as those that can be thought of as *one* thing, and identifies these with sets. At first glance this may appear to be an innocent change in terminology, lacking any deep conceptual significance. However, this change, combined with Cantor's apparent use of Ω (the ordinal sequence) and the $\aleph$–sequence in mathematical argumentation (albeit in proofs that said such sequences cannot be considered sets), and Cantor's renewed emphasis on the difference between the absolute and the transfinite has led some to argue that beginning around 1897, Cantor began to think of absolute infinities/ inconsistent multiplicities as potential infinities. Jané (1995) makes this point explicitly, citing especially Cantor's letters to Hilbert between 1897 and 1899.

I take it that Jané's argument rests particularly on three phrases from Cantor's letters to Hilbert from 2.9.1897 and 6.10.1898 (reprinted in Meschkowski & Nilson, 1991, pp. 390, 393–5, respectively[1])

First is the contrast of "absolutely infinite sets" (i.e. inconsistent multiplicities[2]) to the transfinite sets, of which "it is possible to think without contradiction... of all of their [sic] elements as being together, and consequently, of the set itself as a thing in itself; or again (in other words) if it is possible to think of the set together with the totality of their elements as actually existing." The second important passage says that "the totality of all alephs cannot be conceived as a definite and also completed set" (both of these quotations are from the 1897 letter, translated in Jané, 1995, p. 389).

Note that in both of these passages, it is the possibility of *thinking of* or *conceiving* of an inconsistent multiplicity as a completed object that is questioned. No metaphysical claim is being made. This *thinking of* or *conceiving* can easily be thought of as implicitly including only human thought, as opposed to divine thought. Read in this way, it is still possible for inconsistent multiplicities to exist in *Intellectus Divino*, the Mind of God. This, in turn, is consistent with Cantor's earlier view that the Absolute was beyond human comprehension, but nevertheless actual, due to its existence in the Mind of God.

The third passage, consisting of two paragraphs from the 1898 letter, and paraphrased by Jané (1995, p. 390), says

> that all of the Alephs are not coexistent, cannot be brought together (*zusammengefasst*) as a 'thing in itself', in other words regarded as a completed set.
>
> [...]

[1]The dates referenced by Jané and Meschowski & Nilson for the first letter differ by exactly one month, but it it clear that it is the same letter.

[2]The use of the word *Mengen*, translated as 'sets' here, should not be taken to signify a commitment to the $\aleph$–sequence as a set, but rather a lapse in terminological consistency. This is supported by an endnote in Jané (1995), as well as by Cantor himself in various letters (cf. Meschkowski & Nilson, 1991).

> The *absolute unboundedness* of the set[3] of all Alephs appears as grounds for the impossibility that they can be brought together as a completed thing in itself (Meschkowski & Nilson, 1991, p. 395, my translation, all emphasis original)

At first it may seem that this passage vindicates the reading of Cantor as taking inconsistent multiplicities to be potential, but this is not the only way to read it. That 'all of the Alephs are not coexistent' must be seen only as a contrast with sets, whose members *are* coexsistent. This is natural because Cantor says explicitly that this is meant to mean that all of the alephs cannot be thought of as a completed set. Furthermore the use of the word 'completed' (*fertig*) is likely meant to differentiate from the non-standard use of the word 'set' elsewhere in the passage, so that 'completed set' just means transfinite set.

The second of the above quoted passages can simply be taken to say that it is impossible to consider the $\aleph$–sequence as a single *mathematical* unit. This seems natural, as Cantor goes on to explain why the antinomies are not problematic for his theory[4]. To avoid the antinomies, we need not take inconsistent multiplicities as potential, but only as non-mathematical. One might object to this by saying that, even earlier in this letter, Cantor makes mathematical use of the $\aleph$–sequence, but upon closer inspection we realise that he can be read as only taking it as mathematical in so far as to show that that assumption leads to a contradiction —that if the $\aleph$–sequence is a set, then there must be a cardinal number larger than itself.

4.2. **The Generating Principles.** Perhaps more convincing is the implicit reliance of Cantor's proof of the inconsistency (i.e. non-sethood) of Ω on the generating principles from the *Grundlagen*. The first generating principle for the ordinals says that for any ordinal α there

[3]see previous footnote

[4]Most notable is the Burali-Forti paradox which shows that the collection of all ordinals cannot be a set (cf. Copi, 1958, for an exposition and interesting discussion). The publication of said paradox in 1897 may have been what prompted Cantor to discuss the antinomies, as he did not think that they were applicable to his theory. It is also clear that Cantor was aware of the Burali-Forti paradox, as well as the analogous cases for the cardinals, and the entire set-theoretic universe (essentially Russell's paradox).

exists a next greater ordinal equal to $\alpha + 1$. The second says that for any sequence of ordinals with no greatest element, there is an ordinal greater than all of them, which we call the limit ordinal. Because of these principles Ω, as the system of all ordinals, cannot itself be an ordinal as that would imply that there is an ordinal greater than Ω that is also *in* Ω. In other words, if Ω is a set, then $\Omega < \Omega$, a contradiction.

Jané (1995) argues that this is problematic because, if the two generating principles are presented more formally, the second principle, which Jané takes to be necessary for the derivation of the contradiction, either depends on the independent existence of ordinals, or collapses to circularity.

He mathematises[5] the second principle of generation thus:

> If A is a set of ordinals without largest element, there is a (unique) ordinal β such that (i) $A < \beta$ (i.e., for all $\alpha \in A$, $\alpha < \beta$) and (ii) for no $\gamma : A < \gamma < \beta$. We put $\beta = \lim A$ (p. 395).

Jané rightly points out that this definition relies on an independent determination of which collections of ordinals can be called sets, and concludes that the generating principles must be extra-mathematical, metaphysical principles(*ibid.*). There are two problems with this argument. First, there is no good reason to believe that the generating principles were meant to be purely mathematical, especially given Cantor's platonism. And second, even if we take Jané's mathematisation as faithful to Cantor's intent, we can take the generating principles to be (at least part of) the definition of what is is to be an ordinal/set of ordinals. Ordinals are well-ordered sets that are discovered via the two generating principles. This definition is circular, but not viciously so assuming that we can identify the finite ordinals independently (and accept the domain principle), which would give us, by the second principle of generation, the smallest tranfinite ordinal, ω.

[5]By 'mathematise,' I mean, in this context, the (in this particular case anachronistic) formalisation of Cantor's notions in purely mathematical (as opposed to philosophical) language.

Given the above analysis, there is at worst, an unresolved tension between Cantor's later use of inconsistent multiplicities in seemingly mathematical contexts, and his earlier insistence that the absolute is the domain of God, and unmathematisable. What is lacking in Jané's analysis of this tension is a thorough consideration of Cantor's theology with respect to his conception of the absolute, which will be the focus of the next section.

5. Back to Religion

Given his continued, and even growing connection to the Catholic church later in his career, it seems unlikely that Cantor would have completely given up his conception of the absolute, and therefore inconsistent multiplicities as being actual, i.e. having objective existence. This is especially true given the explicit identification of the absolute with God. If inconsistent multiplicities are meant to be potentially infinite, as Jané would have us believe, then either the identification with God must be thrown out, or there must be some imperfection in God's knowledge or power. The second option goes directly against the Catholic conception of God that Cantor seems to be working with, so either inconsistent multiplicities were no longer the domain of God, or remained actual and unmathematisable despite having some quasi-mathematical content. It is the latter I find more plausible, as it does not attribute a radical change in thought to Cantor. I will thus argue that, taking Cantor's words at face value and appealing to the fact that he never takes a definite stand on the issue in his later writings, combined with epistolary evidence that Cantor's religious convictions never waned, Cantor is unlikely to have ever thought of inconsistent multiplicities as merely potential.

If the above refutation of Jané is reasonably convincing, we need only to note that after 1896, Cantor writes very little else about the nature of inconsistent multiplicities, mentioning them only in contrast to consistent multiplicities, and without explicit appeals to metaphysics or theology[6]. This lack of evidence, cited by Jané (1995), is certainly not evidence of anything, and is thus consistent with the interpretation that Cantor never gave up his realism about the Absolute. Coupled with this is the copious evidence from Cantor's later writings that his religious convictions remained strong into his later years. I will provide what I take to be a representative sample.

Perhaps first and foremost is a short letter to Pope Leo XIII from February 1896 in which he offers the Pope seven copies of the *confessionem fidei* of Francis Bacon, to which Cantor attached particular significance, as well as three copies of the works of Bacon.[7] Cantor then expresses his love for the Pope and the Holy Roman Catholic Church, signing the letter "Your Holiness's humblest and most highly devoted servant" (Meschkowski & Nilson, 1991, p. 383 —my translation, from Meschkowski's translation from the Latin into German).[8]

The closing seems only appropriate for a letter to the Pope, but the importance put on Bacon's confession of faith, as well as the fact that Cantor was sending gifts to the Pope in the first place is strong evidence that his faith wasn't waning, especially as he says that the gifts were meant as token of his love for the Pope and the Church.

On 15.3.1896 Cantor wrote a letter to Father Thomas Esser, a Jesuit priest in Rome. In this letter, Cantor emphasizes the necessary connection between metaphysics and theology, before turning to the subject of his own mathematics, saying first that "Every extension of our insight into what is possible in creation leads necessarily to an extended cognition of God" (translation from Tapp, 2012, p. 9).[9] This principle, elaborated in this letter and elsewhere, can be taken to say that furthering our knowledge of the transfinite gets us closer to an (unobtainable) knowledge of the absolute, which is identified with God.

[6]See Meschkowski & Nilson (1991, pp. 393–5, 409, 433–4) for letters to Hilbert, Dedekind and Jourdain respectively

[7]Frank Jankunis suggested to me that the numbers of copies may be related to the creation story in Genesis, and the Holy Trinity, respectively.

[8]But thanks to Arlin Daniel for double checking this with the original Latin.

[9]Letter reprinted in full in (Tapp, 2005, pp. 307–312.)

This letter is also relevant as an example of Cantor's wider correspondence with theologians between roughly 1894 and 1896 in which he asks that they confirm the acceptability of his theory of the transfinite theologically (Tapp, 2005, §6.4). This is noteworthy not just because of the date (at worst just before the period we are considering), but also because it shows just how important it was to Cantor that his theory be accepted, not just mathematically, but theologically as well. This is further supported by Cantor's apparent pride, and continual citation of the fact that Cardinal Franzelin had not found his transfinite sets, or his identification of the Absolute with the divine, as theologically problematic(Meschkowski, 1967; Dauben, 1990; Tapp, 2005).

The final piece of evidence I will cite here in favour of Cantor's continued religiosity is not from a letter (though there is plenty of such evidence to be found in e.g. Meschkowski & Nilson (1991) or Tapp (2005)), but rather the aforementioned pamphlet *Ex oriente lux*. Despite the fact that Cantor is arguing against a particular point of Catholic dogma, it is thoroughly grounded in the scripture, and should not be taken as anti-Catholic as it was written with an obvious religiosity. The fact that Cantor published this at all is taken by Tapp to be good evidence of Cantor's continued religiosity (Tapp, 2005, p. 193).

These examples make a good case that Cantor was still very religious later in life, which in turn supports my contention that it is unlikely that he would have ever considered inconsistent multiplicities as merely potential, as that would imply an imperfection on the part of God, with whom such infinities were identified in Cantor's earlier writings.

6. Conclusion

In §4 I argued that neither Jané's (1995) argument from Cantor's letters to Hilbert, nor his argument from the mathematisation of the generating principles for ordinals are sufficient to show that Cantor moved from thinking that of the absolutely infinite as actual to thinking of inconsistent multiplicities as potential. In the first case it is merely a matter of a slight variation in interpretation, while in the latter Jané fails to take into account Cantor's platonism, or his view that mathematics, metaphysics and theology are intimately and necessarily related.

I then argued from letters and other late writings of Cantor, that his religious convictions were just as strong late in his career as they were earlier on, which would make it unlikely that Cantor would have ever taken the absolute as potential, as this would involve questioning God's perfection. In combination with the negative result just mentioned, we have good reason to believe that Cantor maintained his realism well past 1897.

I will end by noting that, although it may be impossible to definitely settle the question (without talking to Cantor), the above arguments show that the proposed interpretation of Cantor is more charitable, and nearer the truth than the interpretation of Jané and his supporters.

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